

VALUATION DOMAINS WHOSE PRODUCTS OF FREE MODULES ARE SEPARABLE

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ABSTRACT. It is proved that if R is a valuation domain with maximal ideal P and if R_L is countably generated for each prime ideal L , then R^R is separable if and only if R_J is maximal, where $J = \cap_{n \in \mathbb{N}} P^n$.

When R is a valuation domain satisfying one of the following two conditions:

- (1) R is almost maximal and its quotient field Q is countably generated
- (2) R is archimedean

Franzen proved in [2] that $R^{\mathbb{N}}$ is separable if and only if R is maximal or discrete of rank one. In [3, Theorem XVI.5.4], Fuchs and Salce gave a slight generalization of this result and showed that $R^{\mathbb{N}}$ is separable if and only if R is discrete of rank one, when R is slender. The aim of this paper is to give another generalization of Franzen's result by proving Theorem 8 below. If the maximal ideal P is principal, we get that R^R can be separable when R is neither maximal nor discrete of rank one. This is a negative answer to [3, Problem 59]. For proving his result, Franzen began by showing that each archimedean valuation domain which is not almost maximal, possesses an indecomposable reflexive module of rank 2. We use a similar argument in the proof of Theorem 8. Finally we give an example of a non-archimedean non-slender valuation domain such that $R^{\mathbb{N}}$ is not separable. This is a positive answer to [3, Problem 58].

In the sequel, R is a commutative unitary ring. An R -module whose submodules are totally ordered by inclusion, is said to be **uniserial**. If R is a uniserial R -module, we say that R is a **valuation ring**.

The **R -topology** of R is the linear topology for which each non-zero ideal is a neighborhood of 0. When R is a valuation ring with maximal ideal P and A is a proper ideal, then R/A is Hausdorff in the R/A -topology if and only if $A \neq Pa$, $\forall 0 \neq a \in R$. We say that R is **(almost) maximal** if R/A is complete in the R/A -topology for each (non-zero) proper ideal $A \neq Pa$, $\forall 0 \neq a \in R$.

From now on, R is a valuation domain, P is its maximal ideal and Q is its field of quotients. Let M be an R -module and let N be a submodule. We say that N is a **pure submodule** of M if $rN = rM \cap N$, $\forall r \in R$. Let M be a torsion-free module. We say that M is **separable** if each pure uniserial submodule is a summand. Recall that each element x of M is contained in a pure uniserial submodule U , where U is the inverse image of the torsion submodule of M/Rx by the canonical map $M \rightarrow M/Rx$. Let M be a non-zero R -module. As in [3] we set:

$$M^\# = \{s \in R \mid sM \subset M\}.$$

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Then M^\sharp is a prime ideal. We say that an ideal A is **archimedean** if $A^\sharp = P$.

Proposition 1. *Let $J = \bigcap_{n \in \mathbb{N}} P^n$. Then, R^Λ is separable for each index set Λ if R_J is maximal.*

Proof. If P is not finitely generated then $J = P$. In this case R is maximal, whence R^Λ is separable by [4, Theorem 51]. Suppose now that $P = Rp$ for some $p \in P$. Let U be a pure uniserial submodule of R^Λ . We must prove that U is a summand. First assume that $U^\sharp = P$. Then $pU \neq U$, whence $U = Ru$ for some $u \in U \setminus pU$. If $u = (u_\lambda)_{\lambda \in \Lambda}$, there exists $\mu \in \Lambda$ such that u_μ is a unit because $pU = U \cap pR^\Lambda$. Then in the product R^Λ , the μ th component can be replaced by Ru . So, U is a summand. Now assume that $U^\sharp \subseteq J$. It follows that U is a pure uniserial R_J -submodule of $(R^\Lambda)_J$. Since R_J is maximal, U is a summand of $(R^\Lambda)_J$. Then U is a summand of R^Λ too. \square

From Proposition 1 we deduce the following example which gives a negative answer to [3, Problem 59].

Example 2. Let $R = \mathbb{Z}_p + X\mathbb{Q}[[X]]$, where p is a prime number and \mathbb{Z}_p is the localization of \mathbb{Z} at the prime ideal $p\mathbb{Z}$. Then $J = X\mathbb{Q}[[X]]$, $R/J \cong \mathbb{Z}_p$ and $R_J \cong \mathbb{Q}[[X]]$. It follows that R is neither maximal nor discrete of rank one, but R_J is maximal, whence R^Λ is separable for each index set Λ by Proposition 1. So, [3, Exercise XVI.5.5] is wrong.

To prove Theorem 8 some preliminary results are needed.

If M is an R -module, $\text{Hom}_R(M, R)$ is denoted by M^* and $\lambda_M : M \rightarrow M^{**}$ is the canonical map. We say that M is **reflexive** if λ_M is an isomorphism. An R -module F is **pure-injective** if for every pure exact sequence $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ of R -modules, the following sequence

$$0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$$

is exact. An R -module B is a **pure-essential extension** of a submodule A if A is a pure submodule of B and, if for each submodule K of B , either $K \cap A \neq 0$ or $(A + K)/K$ is not a pure submodule of B/K . We say that B is a **pure-injective hull** of A if B is pure-injective and a pure-essential extension of A . By [3, Chapter XIII] each R -module M has a pure-injective hull and any two pure-injective hulls of M are isomorphic. For any module M , we denote by \widehat{M} its pure-injective hull. If S is a maximal immediate extension of R , then $S \cong \widehat{R}$ by [3, Proposition XIII.5.1]. For each $s \in S \setminus R$, $B(s) = \{r \in R \mid s \notin R + rS\}$ is called the **breadth ideal** of s .

Proposition 3. *Let A be a non-zero archimedean ideal such that $A \neq Pa$ for each $a \in R$. Assume that R/A is not complete in the R/A -topology. Then there exists an indecomposable reflexive module of rank 2.*

Proof. Since R/A is not complete in the R/A -topology, by [3, Lemma V.6.1] there exists $x \in \widehat{R} \setminus R$ such that $A = B(x)$. Let U be a pure uniserial submodule of \widehat{R}/R containing $x + R$ and let M be the inverse image of U by the natural map $\widehat{R} \rightarrow \widehat{R}/R$. Then M is a pure submodule of \widehat{R} . By [3, Example XV.6.1] M is indecomposable. Since M is a pure extension of R by U then $\text{Ext}_R^1(U, R) \neq 0$ and U is torsion-free. Now, we show that $U^\sharp = P$. Let $0 \neq s \in P$. Then $A \subset s^{-1}A$. Let $t \in (R \cap s^{-1}A) \setminus A$. Therefore $x = r + ty$ for some $r \in R$ and $y \in \widehat{R}$. Since

M is a pure submodule of \widehat{R} , we may assume that $y \in M$. By [5, Lemma 1.3], $B(y) = t^{-1}A$. Consequently $y + R \notin sU$ and $U^\# = P$. Since U is a torsion-free module of rank one and $U^\# \neq 0$, U is isomorphic to a proper submodule of Q . So U is isomorphic to an ideal of R . By [2, Proposition 3.3], the proof is complete. \square

Proposition 4. *Assume that R^Λ is separable for an index set Λ . Then $(R_L)^\Lambda$ is separable for each prime ideal L .*

Proof. The assertion is obvious if $L = 0$. Now suppose $L \neq 0$ and let $0 \neq a \in L$. Then $R_L a$ is an ideal contained in L . Let U be a pure uniserial submodule of $(aR_L)^\Lambda$ and let V be the inverse image of the torsion submodule of R^Λ/U by the surjection of R^Λ onto R^Λ/U . Then V is a pure uniserial submodule of R^Λ . Let p be a projection of R^Λ onto V and $q = p|_{(aR_L)^\Lambda}$. For each $s \in R \setminus aR_L$ we have $(aR_L)^\Lambda \subseteq sR^\Lambda$. Thus $\text{Im } q \subseteq sR^\Lambda$. Since $aR_L = \bigcap_{s \in R \setminus aR_L} sR$ we get $\text{Im } q \subseteq (aR_L)^\Lambda$. On the other hand $U \subseteq \text{Im } q$ and the equality holds because U is a pure submodule. It is obvious that $(aR_L)^\Lambda = a(R_L)^\Lambda \cong (R_L)^\Lambda$. Hence $(R_L)^\Lambda$ is separable. \square

Lemma 5. *Let L be a prime ideal of R and let A be a proper ideal of R_L . If R/A is complete in the R/A -topology then R_L/A is also complete in the R_L/A -topology.*

Proof. Let $(a_i + A_i)_{i \in I}$ be a family of cosets of R_L such that $a_i \in a_j + A_j$ if $A_i \subset A_j$ and such that $A = \bigcap_{i \in I} A_i$. We may assume that $A_i \subseteq L$, $\forall i \in I$. So, $a_i + L = a_j + L$, $\forall i, j \in I$. Let $b \in a_i + L$, $\forall i \in I$. It follows that $a_i - b \in L$, $\forall i \in I$. Since R/A is complete in the R/A -topology, $\exists c \in R$ such that $c + b - a_i \in A_i$, $\forall i \in I$. Hence R_L/A is complete in the R_L/A -topology too. \square

Recall that an R -module M is **slender** if for every morphism $f : R^\mathbb{N} \rightarrow M$, there exists $n_0 \in \mathbb{N}$ such that $f(e_n) = 0$, $\forall n \geq n_0$, where $e_n = (\delta_{n,m})_{m \in \mathbb{N}}$. In the proof of Theorem 8 we will use the following result:

Proposition 6. [1, Corollary 21] *Let R be a valuation domain such that Q is countably generated. Then R is slender if and only if R is not complete in the R -topology.*

The following proposition can be easily proved.

Proposition 7. *The following conditions are equivalent:*

- (1) R_L is countably generated for each prime ideal L .
- (2) For each prime ideal L which is the intersection of the set of primes containing properly L there is a countable subset whose intersection is L .
- (3) For each prime ideal L , the quotient field of R/L is countably generated.

Theorem 8. *Assume that R satisfies the equivalent conditions of Proposition 7. Let $J = \bigcap_{n \in \mathbb{N}} P^n$. Then the following conditions are equivalent:*

- (1) R^Λ is separable for each index set Λ ;
- (2) R^R is separable;
- (3) R_J is maximal.

Moreover, if each ideal is countably generated then these conditions are equivalent to: $R^\mathbb{N}$ is separable.

Proof. It is obvious that (1) \Rightarrow (2). By Proposition 1, (3) \Rightarrow (1).

(2) \Rightarrow (3). We must prove that R_J/A is complete in the R_J/A -topology for each ideal A of R_J , where $A \neq Jr$, $\forall 0 \neq r \in R$. By Lemma 5 it is enough to show that R/A is complete in the R/A -topology.

First we assume that A is prime, $A \subset J$. Suppose that R/A is not complete in the R/A -topology. By [3, Lemma XVI.5.3], $(R/A)^{\mathbb{N}}$ is separable. Since R satisfies the conditions of Proposition 7, the quotient field of R/A is countably generated. It follows by Proposition 6 that R/A is slender. By [3, Theorem XVI.5.4] R/A is a discrete valuation domain of rank one. Clearly we get a contradiction. Hence R/A is complete in the R/A -topology. Suppose that $A = rA^{\sharp}$ for some $0 \neq r \in R$ where $A^{\sharp} \subset J$. It is easy to deduce the completeness of R/A from that of R/A^{\sharp} .

Now assume that $A \neq rA^{\sharp}$, $\forall r \in R$. First we show that $R_{A^{\sharp}}/A$ is complete in the $R_{A^{\sharp}}/A$ -topology. By way of contradiction, suppose it is not true. We put $R' = R_{A^{\sharp}}$ and $N^* = \text{Hom}_{R'}(N, R')$ if N is an R' -module. Then A is an archimedean ideal of R' . By Proposition 3 there exists an indecomposable reflexive R' -module M of rank 2. The map $\phi : M^{**} \rightarrow (R')^{M^*}$ defined by $\phi(u) = (u(m))_{m \in M^*}$, $\forall u \in M^{**}$, is a pure monomorphism. Since M^* has the same cardinal as R , $(R')^{M^*}$ is separable by Proposition 4. It follows that M is separable. This contradicts that M is indecomposable.

Now we prove that R/A is complete in the R/A -topology. Let $(a_i + A_i)_{i \in I}$ be a family of cosets of R such that $a_i \in a_j + A_j$ if $A_i \subset A_j$ and such that $A = \bigcap_{i \in I} A_i$. We may assume that $A \subset A_i \subseteq A^{\sharp}$, $\forall i \in I$. We put $A'_i = (A_i)_{A^{\sharp}}$, $\forall i \in I$. We know that $A = \bigcap_{a \notin A} A^{\sharp}a$. Consequently, if $a \notin A$, there exists $i \in I$ such that $A_i \subseteq A^{\sharp}a$, whence $A'_i \subseteq A^{\sharp}a$. It follows that $A = \bigcap_{i \in I} A'_i$. Clearly, $a_i \in a_j + A'_j$ if $A'_i \subset A'_j$. Then there exists $c \in R_{A^{\sharp}}$ such that $c \in a_i + A'_i$, $\forall i \in I$. Since $A'_i \subset R$, $\forall i \in I$, $c \in R$. From $A = \bigcap_{j \in I} A'_j$ and $A \subset A_i$, $\forall i \in I$ we deduce that $\forall i \in I$, $\exists j \in I$ such that $A'_j \subset A_i$. We get that $c \in a_i + A_i$ because $c - a_j \in A'_j \subseteq A_i$ and $a_j - a_i \in A_i$. So, R/A is complete in the R/A -topology.

To prove the last assertion it is enough to observe that M^* is countably generated over R' and consequently M^{**} is isomorphic to a pure R' -submodule of $(R')^{\mathbb{N}}$. \square

Remark 9. In proving that R/A is complete, we use the hypothesis that R satisfies the conditions of Proposition 7 only when A is isomorphic to a prime ideal. In the other case, this result can be obtained with the sole hypothesis that R^R is separable.

So, even if R doesn't satisfy the conditions of Proposition 7 the next proposition holds:

Proposition 10. *Let the notations be as in Theorem 8 and suppose that R^R is separable. Then R satisfies the following conditions:*

- (1) R/L is not slender for each prime ideal $L \subset J$.
- (2) R/A is complete in the R/A -topology for each ideal A which is not isomorphic to a prime ideal and such that $A^{\sharp} \subseteq J$.

The following example gives a positive answer to [3, Problem 58].

Example 11. Let T be a non-discrete archimedean valuation domain which is not complete in the T -topology, let K be its quotient field and let $R = T + XK[[X]]$. Let $L = XK[[X]]$. Then Q and R_L are countably generated. Moreover R is complete in the R -topology because $R_L (\cong K[[X]])$ is maximal and $R/L (\cong T)$ is not complete in the R/L -topology. So, R is non-archimedean, $R^{\mathbb{N}}$ is not separable and R is not slender.

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